GOVERNING EQUATIONS FOR MULTI-LAYERED TSUNAMI WAVES

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ABSTRACT

In order to develop generalized governing equations for multi-layered long wave system, a three layer system was considered. It was anticipated that equations for top layer and bottom layer will be independent on number of intermediate layer(s) and equation for intermediate layer will be generalized depending on number of intermediate layers. However, from derived equations, it is found that only top layer equations are independent of number of intermediate layers; equations for all other layers are dependent on number, extent and density of intermediate layer(s). Momentum and continuity equations for the top layer are exactly same as in the case of earlier developed governing equations for two layered system. Continuity equation for the bottom layer is also exactly same as in the case of two-layered system. Momentum equation for the bottom layer is dependent on extent and density of top layer as well as all intermediate layers. Continuity equation for intermediate layer is affected by levels of immediate bottom layer. Momentum equation for the intermediate layer is affected by extent and density of upper layer(s). Six governing equations, two for each layer are derived from Euler equations of motion and continuity, assuming long wave approximation, negligible friction and interfacial mixing. This paper depicts details derivations of the governing equations.

1. INTRODUCTION

Multi-layered flow is related with many environmental phenomena. Thermally driven exchange flows through doorways to oceanic currents, salt water intrusion in estuaries, spillage of the oil on the sea surface, spreading of dense contaminated water, sediment laden discharges into lakes, generation of lee waves behind a mountain range and tidal flows over sills of the ocean are examples of multi-layered flow. In hydraulics, this type of flow is often termed as gravity current. An extensive review on hydrodynamics of various gravity currents was provided by Simpson, J.E. (1982).

Tsunamis are generated due to disturbances of free surfaces caused by not only seismic fault motion, but also landslide and volcanic eruptions (Imamura and Imteaz, 1995). Tsunami waves are also affected by density differences along the depth of ocean. There are some studies on two-layered long waves or flows in the case of underwater landslide generated tsunamis (Hampton, 1972; Parker, 1982; Harbitz, 1991; Jiang & Leblond, 1992). Imamura & Imteaz (1995) developed a linear numerical model on two-layered long wave flow, which was successfully validated by a rigorous analytical solution. Later the linear model was extended to a non-linear model and effects of non-linearity were investigated (Imteaz & Imamura, 2001).

Madsen et al. (2002) developed a model of multi-layered flow based on Boussinesq-type equations, which are suitable for shallow depth flow. Lynett and Liu (2004) developed another model of multi-layered flow using piecewise integration of Laplace equation for each individual layer and expanded the model for deep water. This paper provides detailed derivation of multi-layered flow equations based on Navier-Stokes equation (Fig. 1).

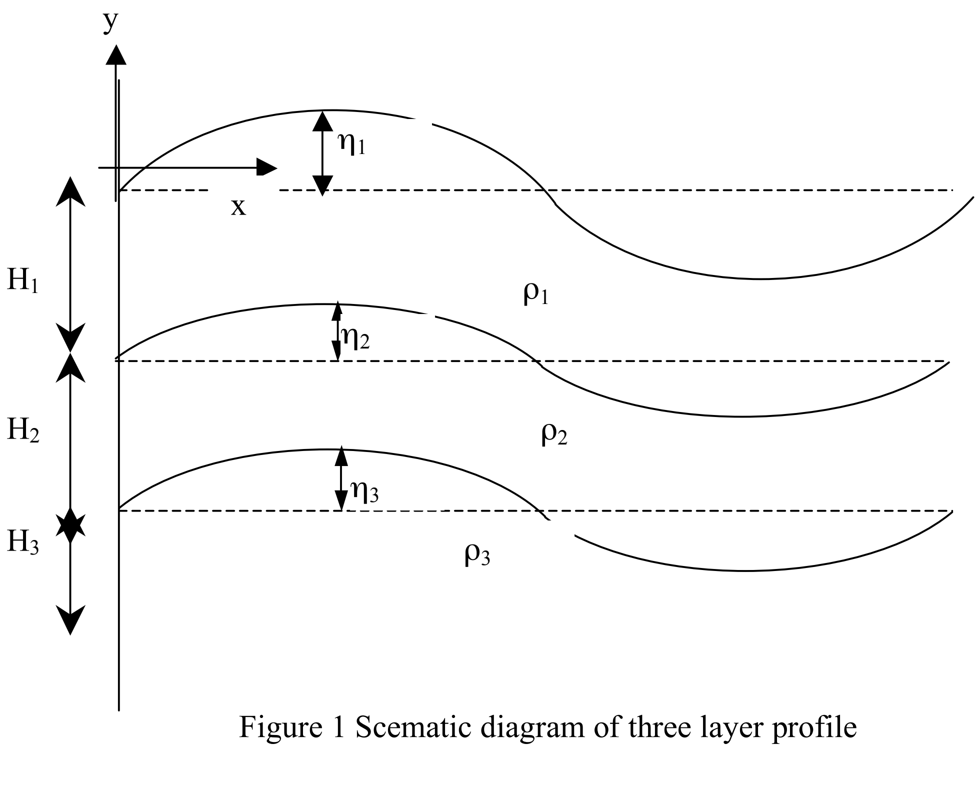

Figure 1 Scematic diagram of three layer profile

2. THEORETICAL CONSIDERATION

Considering two-dimensional case and a three-layered system, continuity equations are as follows:

For upper layer,

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{1}$$

For intermediate layer,

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{2}$$

For bottom layer,

$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} = 0 \tag{3}$$

Momentum equations in X-direction are as follows:

For upper layer,

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho_1} \frac{\partial P_1}{\partial x} \tag{4}$$

For intermediate layer,

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} = -\frac{1}{\rho_2} \frac{\partial P_2}{\partial x} \tag{5}$$

For bottom layer,

$$\frac{\partial u_3}{\partial t} + u_3 \frac{\partial u_3}{\partial x} + v_3 \frac{\partial u_3}{\partial y} = -\frac{1}{\rho_3} \frac{\partial P_3}{\partial x} \tag{6}$$

Where,
u = Uniform velocity over the depth along x-direction
v = Uniform velocity over the depth along y-direction
g = Acceleration due to gravity
P = Hydrostatic pressure of fluid
ρ = Density of fluid
't' represents for time
Subscripts '1', '2' and '3' denotes for upper layer, intermediate layer and bottom layer respectively.

3. BOUNDARY CONDITIONS

Considering a function along any flow streamline, which is constant with time and differentiating it with time, the following boundary conditions have been derived.

At top surface (i.e. at $y = \eta_1$),

$$\frac{\partial \eta_1}{\partial t} + u_1 \frac{\partial \eta_1}{\partial x} = v_1 \tag{7}$$

At interface between top layer and intermediate layer (i.e. at $y = \eta_2 - h_1$),

$$\frac{\partial \eta_2}{\partial t} + u_1 \frac{\partial \eta_2}{\partial x} = v_1 + u_1 \frac{\partial h_1}{\partial x} \tag{8}$$

$$\frac{\partial \eta_2}{\partial t} + u_2 \frac{\partial \eta_2}{\partial x} = v_2 + u_2 \frac{\partial h_2}{\partial x} \tag{9}$$

At interface between bottom layer and intermediate layer (i.e. at $y = \eta_3 - h_1 - h_2$),

$$\frac{\partial \eta_3}{\partial t} + u_2 \frac{\partial \eta_3}{\partial x} = v_2 + u_2 \frac{\partial h_2}{\partial x} + u_2 \frac{\partial h_1}{\partial x} \tag{10}$$

$$\frac{\partial \eta_3}{\partial t} + u_3 \frac{\partial \eta_3}{\partial x} = v_3 + u_3 \frac{\partial h_2}{\partial x} + u_3 \frac{\partial h_1}{\partial x} \tag{11}$$

At bottom (i.e. at $y = -h_3 - h_1 - h_2$),

$$u_3 \left(\frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} + \frac{\partial h_3}{\partial x}\right) = -v_3 \tag{12}$$

Where,

η_1 = Water surface elevation above still water level of layer '1'
η_2 = Water surface elevation above still water level of layer '2'
η_3 = Water surface elevation above still water level of layer '3'
h_1 = Still water depth of layer '1'
h_2 = Still water depth of layer '2'
h_3 = Still water depth of layer '3'

4. GOVERNING EQUATIONS

After integrating continuity equations along the depth, applying some substitution and boundary conditions, following governing equations can be derived considering hydrostatic pressure distribution.

For upper layer- Continuity equation,

$$\frac{\partial M_1}{\partial x} + \frac{\partial(\eta_1 - \eta_2)}{\partial t} = 0 \tag{13}$$

Momentum equation,

$$\frac{\partial M_1}{\partial t} + \frac{\partial(M_1^2 / D_1)}{\partial x} + g D_1 \frac{\partial \eta_1}{\partial x} = 0 \tag{14}$$

For intermediate layer- Continuity equation,

$$\frac{\partial M_2}{\partial x} + \frac{\partial(\eta_2 - \eta_3)}{\partial t} = 0 \tag{15}$$

Momentum equation,

$$\frac{\partial M_2}{\partial t} + \frac{\partial(M_2^2 / D_2)}{\partial x} + g D_2 \left\{ \frac{\alpha_1}{\alpha_2} \left(\frac{\partial \eta_1}{\partial x} + \frac{\partial h_1}{\partial x} - \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_2}{\partial x} - \frac{\partial h_1}{\partial x} \right\} = 0 \tag{16}$$

For lower layer-
Continuity equation,

$$\frac{\partial M_3}{\partial x} + \frac{\partial \eta_3}{\partial t} = 0 \tag{17}$$

Momentum equation,

$$\frac{\partial M_3}{\partial t} + \frac{\partial(M_3^2 / D_3)}{\partial x} + g D_3 \left\{ \alpha_1 \left(\frac{\partial \eta_1}{\partial x} + \frac{\partial h_1}{\partial x} - \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_3}{\partial x} - \frac{\partial h_1}{\partial x} - \frac{\partial h_2}{\partial x} + \alpha_2 \left(\frac{\partial \eta_2}{\partial x} + \frac{\partial h_2}{\partial x} - \frac{\partial \eta_3}{\partial x} \right) \right\} = 0 \tag{18}$$

Where,
η_1 = Water surface elevation above still water level of layer '1'
η_2 = Water surface elevation above still water level of layer '2'
η_3 = Water surface elevation above still water level of layer '3'
$D_1 = \eta_1 + h_1 - \eta_2$, $D_2 = h_2 + \eta_2 - \eta_3$, $D_3 = h_3 + \eta_3$, $\alpha_1 = \rho_1/\rho_3$, $\alpha_2 = \rho_2/\rho_3$
h_1 = Still water depth of layer '1'
h_2 = Still water depth of layer '2'
h_3 = Still water depth of layer '3'

$$M_1 = \int_{-h_1+\eta_2}^{\eta_1} u_1 \, dy, \quad M_2 = \int_{-h_1-h_2+\eta_3}^{-h_1+\eta_2} u_2 \, dy, \quad M_3 = \int_{-h_1-h_2-h_3}^{-h_1-h_2+\eta_3} u_3 \, dy$$

Above derived equations can be further simplified neglecting non-linear terms, assuming small amplitude waves (i.e. $\eta << h$ and $D \approx h$) and no variations of 'h' along x direction (i.e. $\partial h/\partial x=0$).
Final simplified equations are as follows:

For upper layer - Continuity equation,

$$\frac{\partial M_1}{\partial x} + \frac{\partial(\eta_1 - \eta_2)}{\partial t} = 0 \qquad (19)$$

Momentum equation,

$$\frac{\partial M_1}{\partial t} + g\, h_1 \frac{\partial \eta_1}{\partial x} = 0 \qquad (20)$$

For intermediate layer - Continuity equation,

$$\frac{\partial M_2}{\partial x} + \frac{\partial(\eta_2 - \eta_3)}{\partial t} = 0 \qquad (21)$$

Momentum equation,

$$\frac{\partial M_2}{\partial t} + g\, h_2 \left\{ \frac{\alpha_1}{\alpha_2} \left(\frac{\partial \eta_1}{\partial x} - \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_2}{\partial x} \right\} = 0 \qquad (22)$$

For lower layer - Continuity equation,

$$\frac{\partial M_3}{\partial x} + \frac{\partial \eta_3}{\partial t} = 0 \qquad (23)$$

Momentum equation,

$$\frac{\partial M_3}{\partial t} + g\, h_3 \left\{ \alpha_1 \left(\frac{\partial \eta_1}{\partial x} - \frac{\partial \eta_2}{\partial x} \right) + \frac{\partial \eta_3}{\partial x} + \alpha_2 \left(\frac{\partial \eta_2}{\partial x} - \frac{\partial \eta_3}{\partial x} \right) \right\} = 0 \qquad (24)$$

From the derived equations, it is found that momentum equation for upper layer is not affected by the properties of adjacent layer (layer underneath). However, continuity equation of upper layer is affected by surface elevation of intermediate layer. Continuity equation for intermediate layer is affected by the surface elevation of bottom layer. Momentum equation for intermediate layer is affected by density and spatial change of surface elevation of upper layer. Continuity equation for bottom layer is not affected by either uppermost layer or intermediate layer. However, momentum equation of bottom layer is affected by densities and spatial changes in surface elevations of all the layers above it. Properties of all these equations are outlined in Table 1.

TABLE 1 Affect of a particular layer for a particular equation from adjacent layer(s)

<table>
<tr><th>Layer</th><th>Equation</th><th>Effect from adjacent layer</th></tr>
<tr><td rowspan="2">Upper layer</td><td>Continuity</td><td>• Water surface elevation of intermediate layer</td></tr>
<tr><td>Momentum</td><td>• Not affected</td></tr>
<tr><td rowspan="2">Intermedi-ate layer</td><td>Continuity</td><td>• Water surface elevation of bottom layer</td></tr>
<tr><td>Momentum</td><td>• Spatial change of water surface elevation of upper layer
• Density of upper layer</td></tr>
<tr><td rowspan="2">Bottom layer</td><td>Continuity</td><td>• Not affected</td></tr>
<tr><td>Momentum</td><td>• Spatial change of water surface elevation of upper layer
• Spatial change of water surface elevation of intermediate layer
• Density of upper layer
• Density of intermediate layer</td></tr>
</table>

Differentiating continuity equations with 't' and momentum equations with 'x' and then subtracting each from other, three different equations (one for each layer) can be deduced as follows:

For upper layer-

$$\frac{\partial^2 \eta_1}{\partial t^2} - \frac{\partial^2 \eta_2}{\partial t^2} - gh_1 \frac{\partial^2 \eta_1}{\partial x^2} = 0 \tag{25}$$

For intermediate layer-

$$\frac{\partial^2 \eta_2}{\partial t^2} - \frac{\partial^2 \eta_3}{\partial t^2} - gh_2 \left[\frac{\alpha_1}{\alpha_2} \left(\frac{\partial^2 \eta_1}{\partial x^2} - \frac{\partial^2 \eta_2}{\partial x^2} \right) + \frac{\partial^2 \eta_2}{\partial x^2} \right] = 0 \tag{26}$$

For lower layer-

$$\frac{\partial^2 \eta_3}{\partial t^2} - gh_3 \left[\alpha_1 \left(\frac{\partial^2 \eta_1}{\partial x^2} - \frac{\partial^2 \eta_2}{\partial x^2} \right) + \frac{\partial^2 \eta_3}{\partial x^2} + \alpha_2 \left(\frac{\partial^2 \eta_2}{\partial x^2} - \frac{\partial^2 \eta_3}{\partial x^2} \right) \right] = 0 \tag{27}$$

Substituting Equation (26) to Equation (25) and rearranging,

$$\frac{\partial^2 \eta_1}{\partial t^2} - gh_1 (1 + \alpha_3 \beta_1) \frac{\partial^2 \eta_1}{\partial x^2} - gh_2 (1 - \alpha_3) \frac{\partial^2 \eta_2}{\partial x^2} - \frac{\partial^2 \eta_3}{\partial t^2} = 0 \tag{28}$$

Substituting Equation (27) to Equation (26) and rearranging,

$$\frac{\partial^2 \eta_2}{\partial t^2} - gh_2\left[(1-\alpha_3)+\beta_2(\alpha_2-\alpha_1)\right]\frac{\partial^2 \eta_2}{\partial x^2} - gh_3\left(\alpha_1\frac{\partial^2 \eta_1}{\partial x^2} + \frac{\partial^2 \eta_3}{\partial x^2} - \alpha_2\frac{\partial^2 \eta_3}{\partial x^2}\right) = 0 \qquad (29)$$

In Equations 28 & 29, $\alpha_3=\rho_1/\rho_2$, $\beta_1=h_2/h_1$ and $\beta_2=h_3/h_2$. From Equations 27, 28 & 29 wave celerity for each layer can be defined as:

$$C_1 = \sqrt{gh_1(1+\alpha_3\beta_1)}$$
$$C_2 = \sqrt{gh_2(1+\beta_2(\alpha_2-\alpha_1))}$$
$$C_3 = \sqrt{gh_3(1-\alpha_2)}$$

5. CHARACTERISTICS OF MULTI-LAYER EQUATIONS

It is found that the developed governing equations are complicated having influence from upper and/or lower layer flow. However, wave celerities for three different layers are derived as shown above. Wave celerity of a particular layer can be compared with the wave celerity of other layer considering reasonable values of the associated parameters.

Figure 2 shows the relationship of normalized celerity of upper layer with β_1 for different α_3 values. Upper layer celerity increases with the increase of both the β_1 and α_3 values having a power relationship.

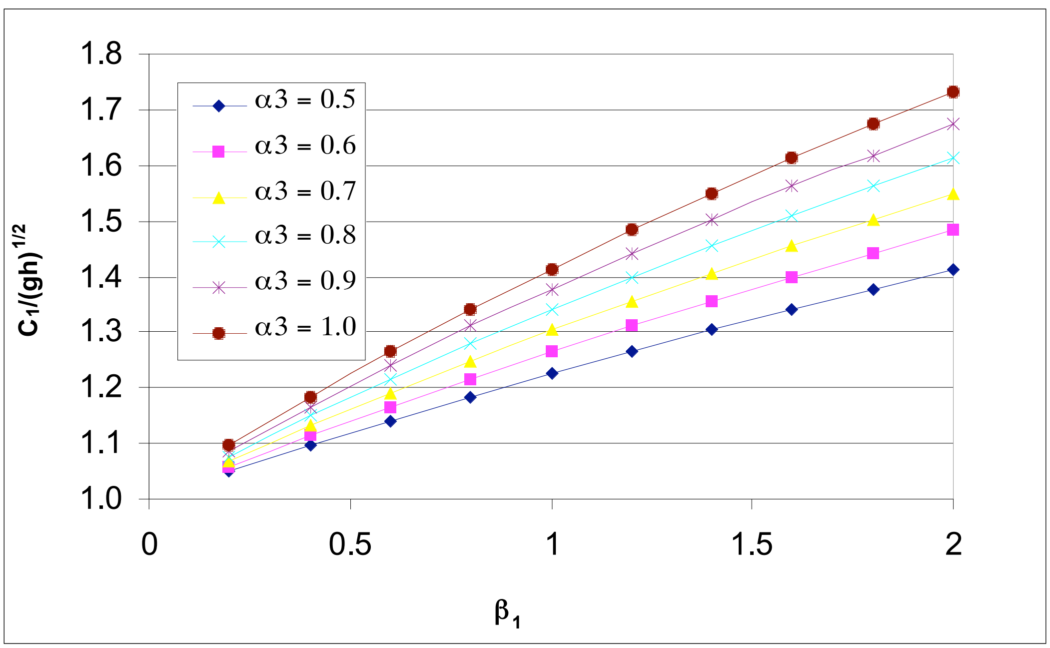

Figure 2 Relationship of normalized celerity of upper layer with β_1

Figure 3 shows the relationship of normalized celerity of intermediate layer with β_2 for different $(\alpha_2 - \alpha_1)$ values. Intermediate layer celerity increases with the increase of both the β_2 and $(\alpha_2 - \alpha_1)$ values having a power relationship. However, normalized celerity of lower layer depends only on the ratio of density of intermediate layer to the density of lower layer (i.e. $\alpha_2 = \rho_2/\rho_3$).

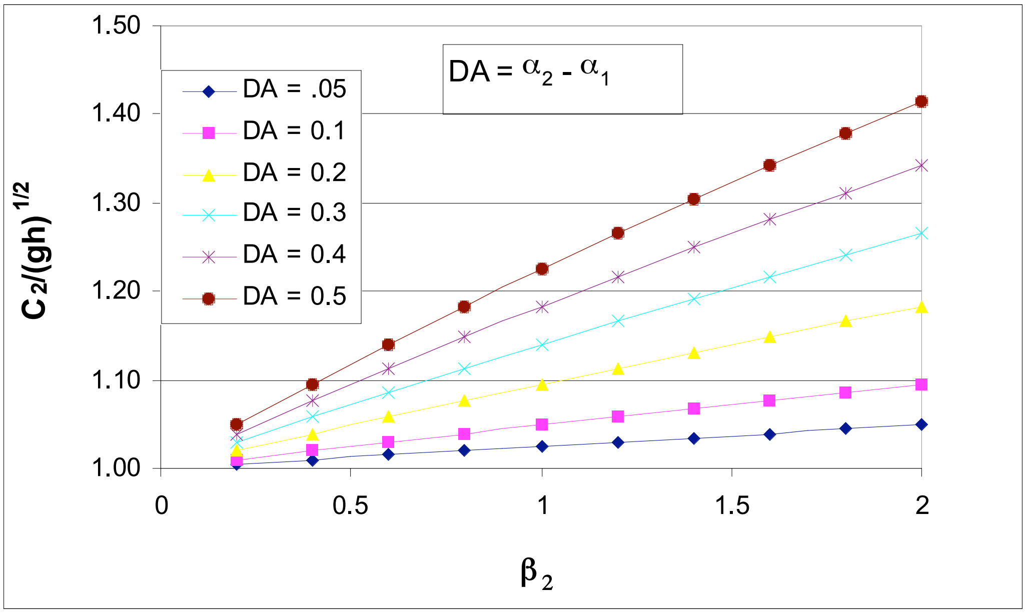

Figure 3 Relationship of normalized celerity of intermediate layer with β_2

Celerity decreases rapidly with the increase of α_2 values as shown in Figure 4.

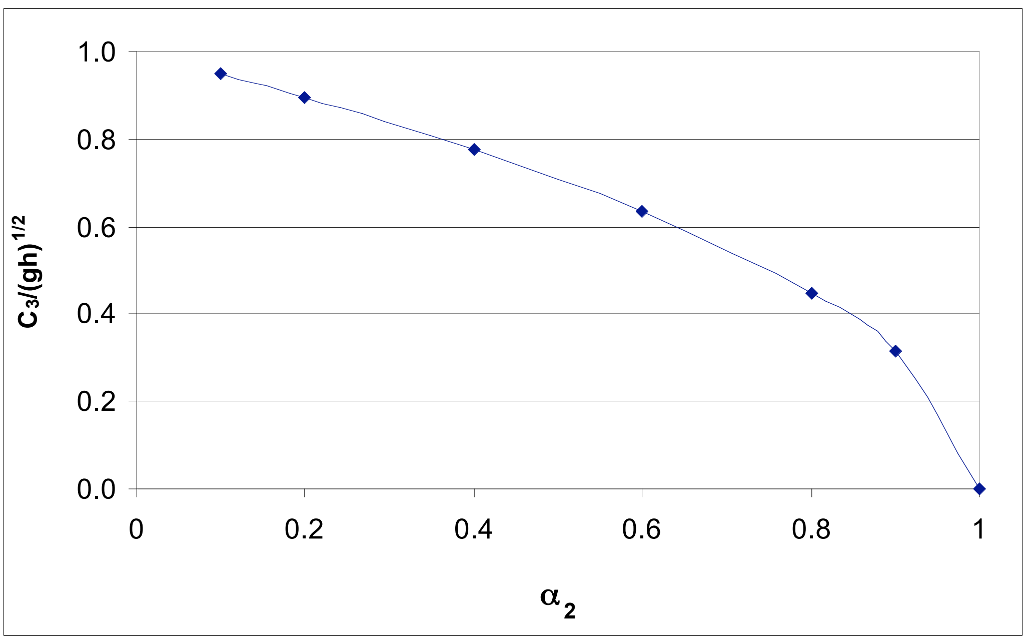

Figure 4 Relationship of normalized celerity of lower layer with α_2

Figure 5a shows the relationship of C_1/C_2 with $(\alpha_2 - \alpha_1)$ for different values of α_3 and Figure 5b shows the relationship of C_1/C_2 with α_3 for different $(\alpha_2 - \alpha_1)$ values. It is shown that C_1/C_2 decreases significantly with the increase of $(\alpha_2 - \alpha_1)$. However, C_1/C_2 increases with the increase of α_3.

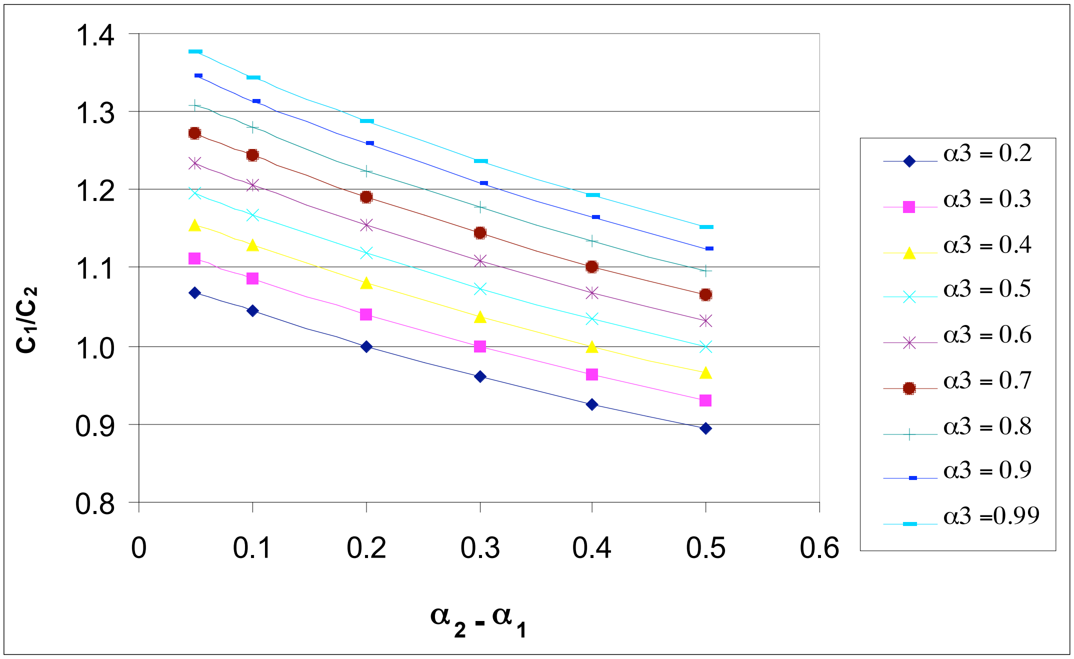

Figure 5a Relationship of C_1/C_2 with $(\alpha_2 - \alpha_1)$

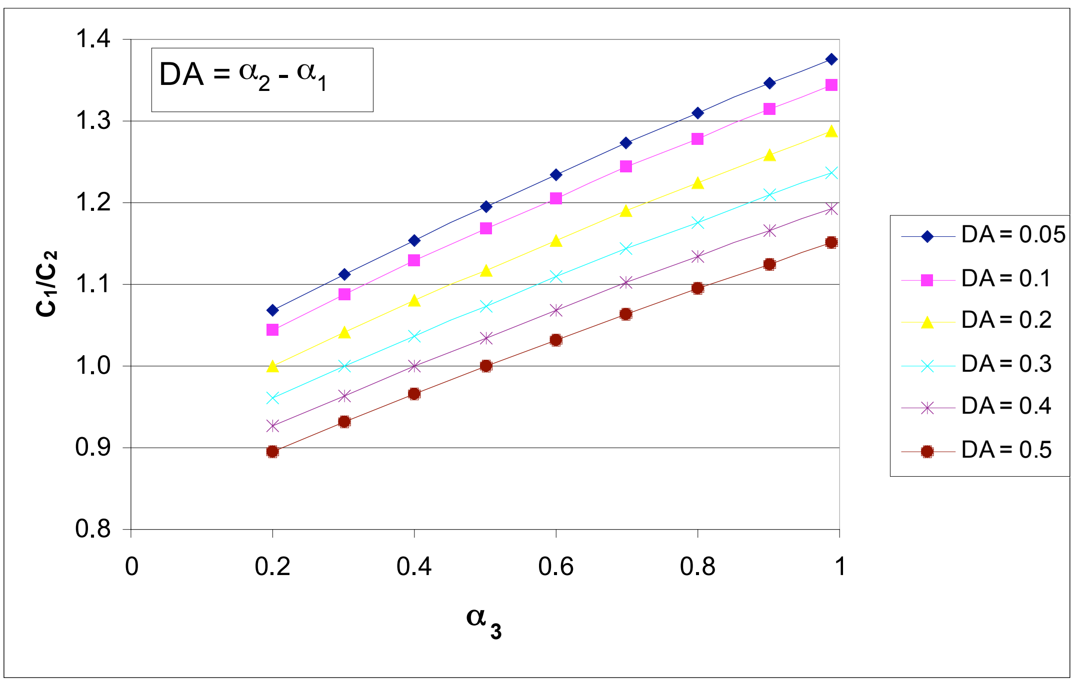

Figure 5b Relationship of C_1/C_2 with α_3

Figure 6a shows the relationship of C_2/C_3 with $(\alpha_2 - \alpha_1)$ for different values of α_2 and Figure 6b shows the relationship of C_2/C_3 with α_2 for different $(\alpha_2 - \alpha_1)$ values. It is shown that C_2/C_3 increases almost linearly with the increase of $(\alpha_2 - \alpha_1)$. However, for a particular value of $(\alpha_2 - \alpha_1)$, C_2/C_3 increases almost exponentially with the increase of α_2.

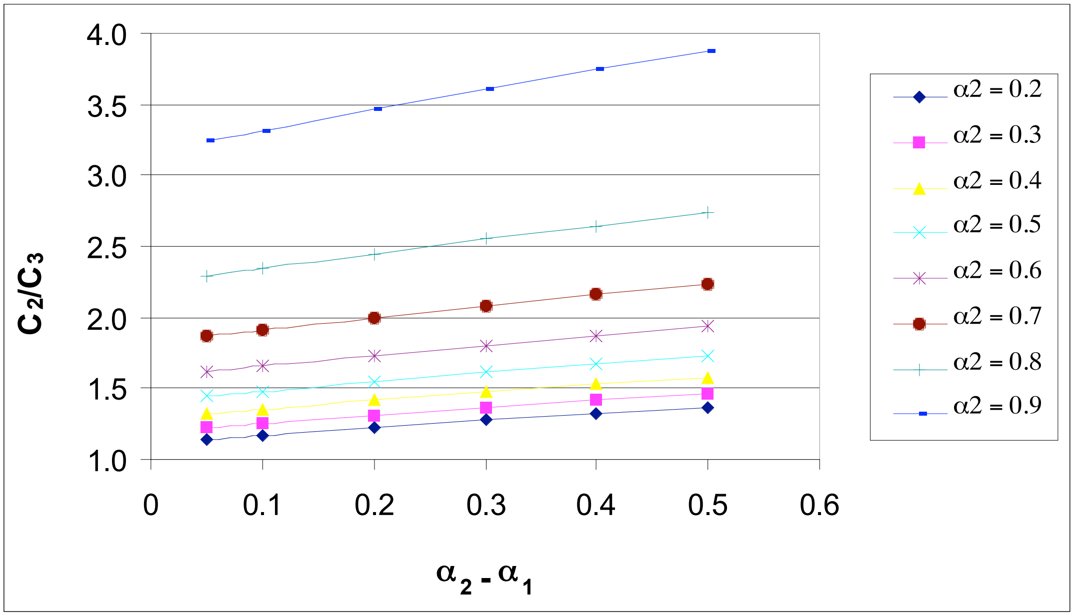

Figure 6a Relationship of C_2/C_3 with $(\alpha_2 - \alpha_1)$

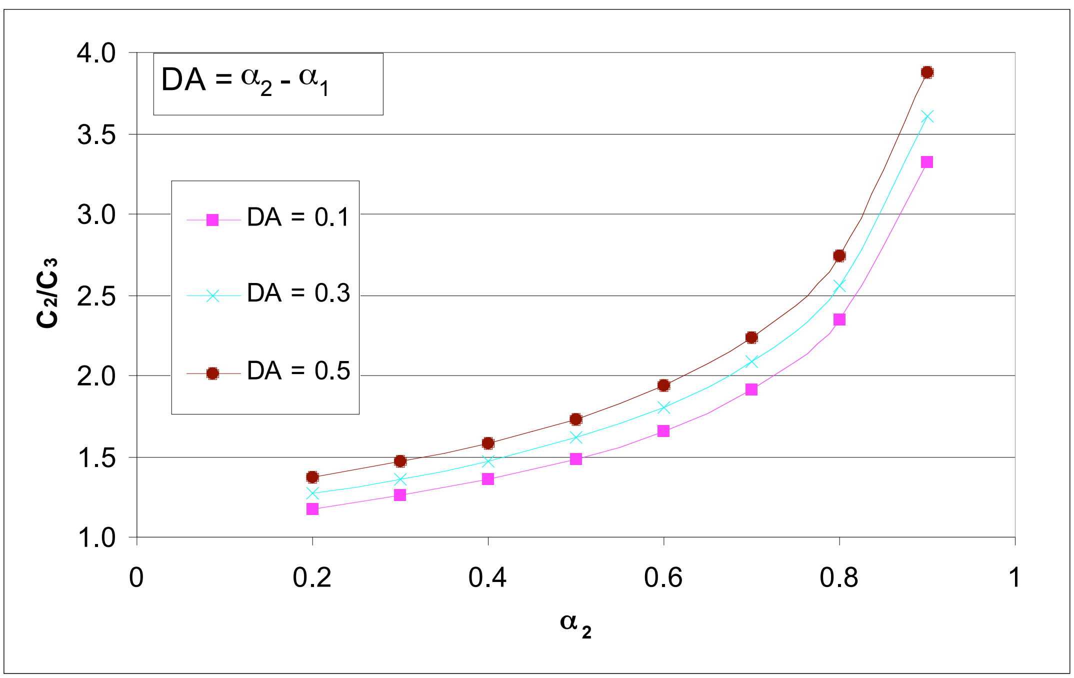

Figure 6b Relationship of C_2/C_3 with α_2

Figure 7a shows the relationship of C_1/C_3 with α_3 for different values of α_2 and Figure 7b shows the relationship of C_1/C_3 with α_2 for different α_3. It is shown that C_1/C_3 increases almost linearly with the increase of α_3. However, for a particular value of α_3, C_1/C_3 increases almost exponentially with the increase of α_2.

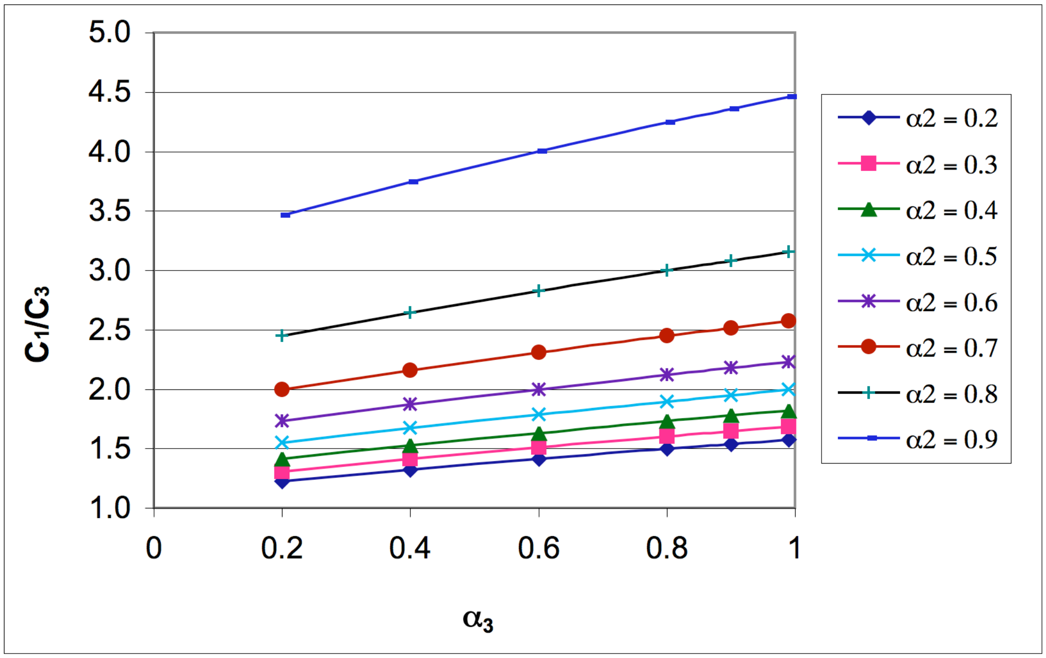

Figure 7a Relationship of C_1/C_3 with α_3

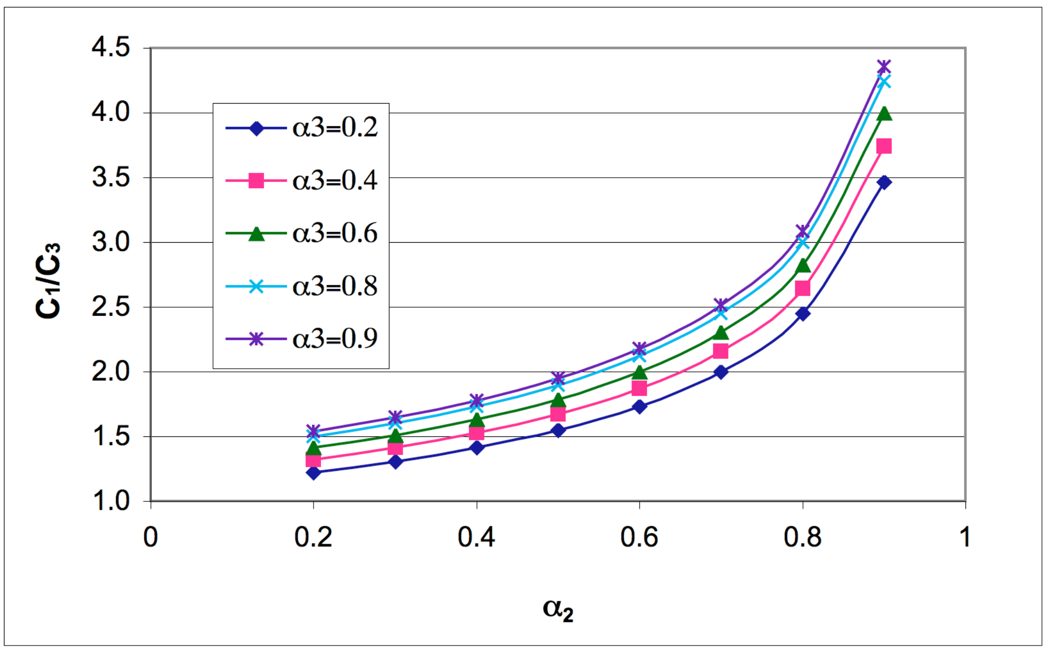

Figure 7b shows the relationship of C_1/C_3 with α_2

Figure 8a shows the relationship of C_2/C_3 with α_2 for different values of β and Figure 8b shows the relationship of C_2/C_3 with $(\alpha_2 - \alpha_1)$ for different values of α_2. It is shown that C_2/C_3 increases almost exponentially with the increase of α_2. However, for a particular value of α_2, C_2/C_3 increases almost linearly with the increase of $(\alpha_2 - \alpha_1)$.

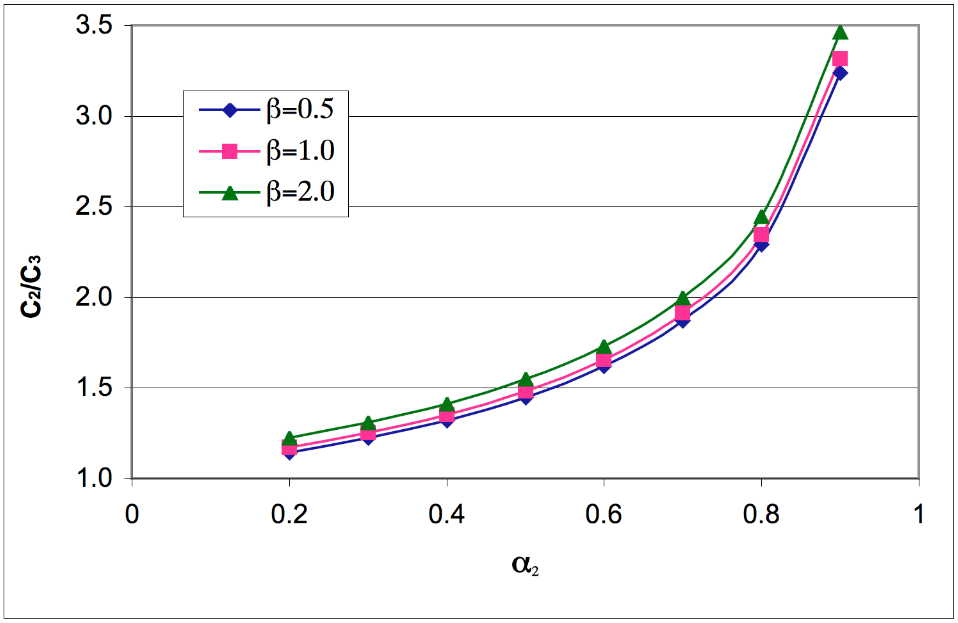

Figure 8a shows the relationship of C_2/C_3 with α_2

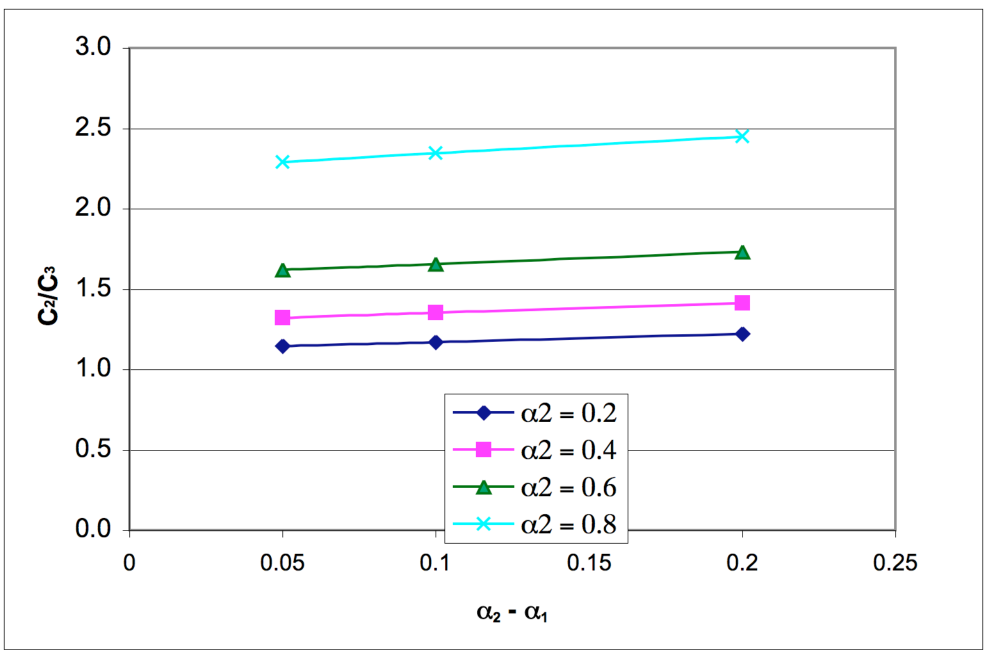

Figure 8b shows the relationship of C_2/C_3 with $(\alpha_2 - \alpha_1)$

6. CONCLUSION

Governing equations for three layered tsunami waves were developed transforming Navier-Stoke equations using proper boundary conditions and long wave approximation. Developed equations for three layers can be easily transformed to equations of any number of layers, considering the effect(s) of adjacent layer(s). Eventually three separate equations (one for each layer) were derived. From these derived equations wave celerity for each layer wave was extracted and presented. Characteristics of wave celerity for each layer were discussed in relation to relative density and/or relative layer depth. In regards to effect of adjacent layer on a particular layer, it is found that the upper layer wave equation is only affected by water surface elevation of intermediate layer. However, intermediate layer equation is affected by water surface elevation of bottom layer, spatial change of elevation of upper layer and density of upper layer. Bottom layer equation is affected by spatial changes and densities of all the layers above this layer. A generalized system of governing equations, which will be able to calculate wave properties for any number of layers, can be developed from these equations. A numerical finite difference model of developed equations will be developed and simulated. For numerical simulations, as Staggered Leap-Frog scheme produced very good results for a two-layer long wave system, it expected that same numerical scheme would be able to produce good results for a multi-layered system as well.

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NOTATIONS

□ = Density of fluid
M = Discharge per unit width of flow
□ = Water surface elevation above still water level
D = Total depth of layer
□ = Ratio of density of upper layer fluid to lower layer fluid
x = Distance along downstream direction
y = Distance perpendicular to x-direction
u = Uniform velocity over the depth along x-direction
v = Uniform velocity along y-direction
g = Acceleration due to gravity
P = Hydrostatic pressure of fluid
□ = Ratio of depths of lower layer to upper layer
C = Wave celerity