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BENCHMARK SOLUTIONS FOR TSUNAMI WAVE FRONTS AND RAYS. PART 1: SLOPING BOTTOM TOPOGRAPHY

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ABSTRACT

In this paper, the kinematics of tsunami wave rays and fronts over an uneven bottom are studied. A formula for the wave height along a ray tube is obtained. An exact analytical solution for wave rays and fronts over a sloping bottom is derived. This solution makes possible to determine a tsunami wave height in an area with a sloping bottom from the initial source in the ray approximation. The distribution of wave-height maxima calculated in an area with a sloping bottom is compared to that obtained with a shallow-water model.

Keywords: *tsunami, wave kinematics, wave ray and front, exact solutions for sloping bottom*

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1. SOME FEATURES OF THE LONG WAVE PROPAGATION

Tsunami waves usually generated by vertical displacements of large ocean bottom areas belong to a class of long waves whose length is at least ten times greater than the depth. The propagation of such waves in a deep ocean, where the wave height is usually two orders smaller than the depth, is described by a linear system of differential shallow-water equations (Stocker, 1957). The validity of this description has many times been verified in practice. In the one-dimensional case without external forces (except for the gravity) these equations can be written down in the following form:

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0 \tag{1.1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial (Du)}{\partial x} = 0 \tag{1.2}$$

Here u is the horizontal water flow velocity in the wave, η is the water surface height above an unperturbed level, g is the acceleration of gravity, and D is the depth. It follows from the shallow-water equations that the wave velocity does not depend on its length, and is determined by the Lagrange formula (Stocker, 1957):

$$c = \sqrt{gD} \tag{1.3}$$

This formula is of fundamental importance for the kinematics of long waves (in particular, tsunamis). The wave front may be defined as interface between the undisturbed water (the height η and velocity components are zero) and the water area, where the perturbation from the source has already arrived at the time instant ($\eta \neq 0$). To describe the tsunami wave dynamics in the coastal zone where the tsunami amplitude increases and the depth decreases, we use the nonlinear shallow water model (Marchuk et al, 1983) in which the wave propagation velocity is expressed by the formula

$$c = \sqrt{g(D+\eta)} \tag{1.4}$$

Let us note that the front and wave crest velocities are somewhat different. However, the crest, where the water surface height reaches a maximum along the entire wavelength, gradually catches up with the front. When the crest passes the front, the wave breaks. If a wave propagates in a deep ocean this effect is weak even if it passes the entire water area of the Pacific Ocean. In what follows, when considering the kinematics of long wave propagation velocity means wave front velocity, which, according to (1.3), does not depend on the wave parameters and is solely defined by the ocean depth at the place, where the wave currently is. The fact that the front propagation velocity of a tsunami wave does not depend on its amplitude and length is determinative for behavior of such waves in water areas with an uneven bottom.

The properties of the system of shallow water equations and peculiarities of the physical process of tsunami wave propagation make possible to obtain estimates for the parameters of the flow in this wave that can be used to determine the tsunami height. Specifically, the fact that the horizontal velocity in the long wave motion is constant in the entire bulk of water (from surface to bottom) is taken into account (Stocker, 1957). All relations to be obtained in this Section are valid for the linearized shallow water equations. These relations are also accurate up to second order for tsunami propagation described by the nonlinear shallow water equations in the deep ocean, where the depth is two orders greater than the wave amplitude. First let us obtain an approximate formula for the horizontal flow velocity u in a moving tsunami wave if the depth is equal to D. Such a relation can be explicitly obtained from the linearized shallow water equations (1.1) and (1.2). We know that in this model the wave velocity is determined by the Lagrange formula (1.3). Let a running wave be represented as a harmonic function

$$\eta = a \times \cos(kx - \omega t), \tag{1.5}$$

which describes a wave of height a running in the direction of increasing the coordinate x with the velocity $c = \omega/k$. Substituting the expression for water surface displacement (1.5) into equation (1.1), we have

$$\frac{\partial u}{\partial t} = gka\sin(kx - \omega t). \tag{1.6}$$

Integrating both sides of (1.6) with respect to t, we obtain the following relation between the flow velocity in the wave and its amplitude and depth:

$$u = \int \frac{gk}{\omega} a\sin(kx - \omega t)d(kx - \omega t) = \frac{g}{c}(a\cos(kx - \omega t)) = \frac{g}{\sqrt{gD}}\eta = \eta\sqrt{\frac{g}{D}}. \tag{1.7}$$

Thus, in a harmonic wave of the form of (1.5) the water flow velocity is defined by formula (1.7). However, since the process is linear, formula (1.7) will be valid for any long wave that can be represented as superposition of harmonic waves with different frequencies and is a solution to the system of linear differential shallow water equations (1.1) and (1.2). For the quasilinear system of shallow water equations, where the wave front and crest velocities are somewhat different and determined by (1.4), the horizontal flow velocity in a moving wave has the form

$$u = \eta\sqrt{\frac{g}{D + \eta}}, \tag{1.8}$$

where η is the wave height, D is the depth, and g is the acceleration of gravity. An expression for the kinetic energy of a propagating one-dimensional tsunami wave with allowance for formula (1.8) is as follows:

$$E_K = \int_0^L \frac{\rho u^2}{2}(D+\eta)dx = \int_0^L \frac{\rho\eta^2}{2}\frac{g}{(D+\eta)}(D+\eta)dx = \int_0^L \frac{\rho\eta^2 g}{2}dx \quad , \tag{1.9}$$

where L is the wavelength, and ρ is the liquid density. Let us also write down an expression for the potential energy assuming that the potential energy of a quiet liquid is zero,

$$E_P = \int_0^L \frac{\rho g\eta^2}{2}dx \tag{1.10}$$

A comparison of the integrands in the formulas for the kinetic and the potential energies of a running tsunami wave show their identical equality. Hence, in any segment of the wave the kinetic energy is equal to the potential energy.

Now, using formula (1.7), we can find an approximate relation between the height of a plane (one-dimensional) tsunami wave and depth. Let at a point $x = x_1$ the depth be equal to D_1 and the mareogram (the ocean surface elevation versus time) of a one-dimensional tsunami wave be expressed by the function $\eta_1(t)$ ($t = 0, T_1$). Taking into account the relation for the wave propagation velocity (1.3) and the fact that the water flow velocity of a long wave is constant in the entire water layer, the potential energy of a wave passing through the cross-section $x = x_1$ can be written down in the form

$$E_P = \int_0^{T_1} \frac{\rho g\eta_1^2(t)}{2} \times \sqrt{gD_1}\,dt \quad . \tag{1.11}$$

Let the wave in question reach a point x_2, the depth be equal to D_2, and the mareogram of the wave at the point x_2 be expressed by a function $\eta_2(t)$. However, the wave period has not changed and remains equal to T_1. It follows from the fact that, naturally, each wave segment passes the same way between the points x_1 and x_2 during the same time interval (no matter how the depth D_1 changes for D_2). Since the total energy of the wave remains constant and the potential energy of a moving wave always constitutes half the total energy (as is shown), the following integral equality is valid for the potential energy of the wave as it passes the cross-sections $x = x_1$ and
$x = x_2$:

$$\int_0^{T_1} \frac{\rho g\eta_1^2(t)}{2} \times \sqrt{gD_1}\,dt = \int_0^{T_1} \frac{\rho g\eta_2^2(t)}{2} \times \sqrt{gD_2}\,dt \quad . \tag{1.12}$$

If the process of tsunami propagation is linear (as in the case of the deep ocean where the wave height is two orders lower than the depth), integral equality (1.12) is transformed to an approximate equality of the integrands (as is shown above). Therefore, after some simplifications we have

$$\eta_1^2(x) \times \sqrt{D_1} \approx \eta_2^2(x) \times \sqrt{D_2} \tag{1.13}$$

Hence, we have the approximate formula for the wave amplitude at the depth D_2:

$$\eta_2(x) \approx \eta_1(x)\sqrt[4]{\frac{D_1}{D_2}} \quad . \tag{1.14}$$

It is the well-known Green's formula describing a height variation of a long wave over an uneven bottom in the one-dimensional case. Thus, as a plane wave propagates from the deep ocean to a shallow-water shelf, its height increases proportional to the fourth root of the ratio between the initial and current depths (formula (1.14)). If a wave is not plane, its amplitude varies not only due to the non-constant depth, but also as a result of wave refraction (that is, the wave-front line transformation).

Let us consider a simple case of an initially circular tsunami wave propagating in an area of constant depth. In this case, according to Lagrange's formula (1.3), the wave front is always circular in shape with a constantly increasing radius, and the wavelength remains constant. It is clear that since the wave front extends, the amplitude steadily decreases. Let us use the energy conservation law to estimate the degree of this decreasing behavior. Let at some time instant the radius of the circular wave front be equal to R_1, at some other time, to R_2, and the lengths of the wave-front circles (arcs of the circle) be L_1 and L_2, respectively. Let us write down the wave potential energy taking into account the fact that the wave parameters are the same along the entire wave-front circle as follows:

$$E_P = \int_0^{L_1}\int_0^{\lambda_1} \frac{\rho g \eta_1^2}{2} d\lambda dl = \int_0^{\lambda_1} \frac{\rho g \eta_1^2}{2} L_1 d\lambda = \int_0^{L_2}\int_0^{\lambda_1} \frac{\rho g \eta_2^2}{2} d\lambda dl = \int_0^{\lambda_1} \frac{\rho g \eta_2^2}{2} L_2 d\lambda \quad . \tag{1.15}$$

Here λ_1 is the wavelength. In the case of a linear wave, integral equality (1.15) reduces to the equality of the integrands

$$\frac{\rho g \eta_1^2}{2} 2\pi R_1 = \frac{\rho g \eta_2^2}{2} 2\pi R_2 \qquad \eta_2 = \eta_1 \sqrt{\frac{R_1}{R_2}} \quad . \quad . \tag{1.16}$$

Thus, due to the cylindrical propagation the tsunami wave height decreases inversely proportional to the square root of the circular front radius or the wave front length.

In general, the kinematics of propagation of perturbations in various media is described by the eikonal equation (Romanov, 1984), which has the following form in the two-dimensional case:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{v^2(x, y)} \quad , \tag{1.17}$$

where v (x, y) specifies the velocity distribution in the medium. If the function f (x, y) is a solution to the eikonal equation (1.17), the wave front at the time T is described by the equation

$$f(xylem)=T, \tag{1.18}$$

and the equation $f(x, y) = 0$ specifies the perturbation source location or the initial wavefront line (the tsunami source boundary).

A concept of wave ray one of whose properties is the orthogonality to the wave-front line at any time is introduced in (Romanov, 1984). Along wave rays, a perturbation propagates from a source to other points of the medium in the least time. This means that wave rays are the fastest routes. Between the two closely spaced wave rays (in a ray tube), the wave energy remains constant (Romanov, 1984). Therefore, for a wave segment in a ray tube, formula (1.16) can be rewritten in the form

$$\eta_2 = \eta_1 \sqrt{\frac{L_1}{L_2}}, \tag{1.19}$$

where L_1 and L_2 are the widths of the ray tube (the length of the wave-front line segment inside the ray tube) at the initial and current time moments of wave propagation.

2. EXACT ANALYTICAL FORMULAS FOR WAVE-RAY TRACES ABOVE THE SLOPING BOTTOM

An exact mathematical formula for a wave ray trajectory over a sloping bottom can be found from the laws of geometrical optics. Consider a two-dimensional water area where the depth and the wave propagation velocity vary only in one direction. In this case we can use Snell's law for the wave ray refraction angle in a medium with varying optical conductivity (Sabra, 1981). According to this law, if in a two-dimensional conducting medium a ray comes at the angle of incidence α_1 to the horizontal line (Fig. 2.1), and the conductivity (propagation velocity of a signal) changes from b_1 to b_2, after passing the interface boundary its direction α_2 changes according to the formula

$$\frac{\sin(\alpha_1)}{b_1} = \frac{\sin(\alpha_2)}{b_2}. \tag{2.1}$$

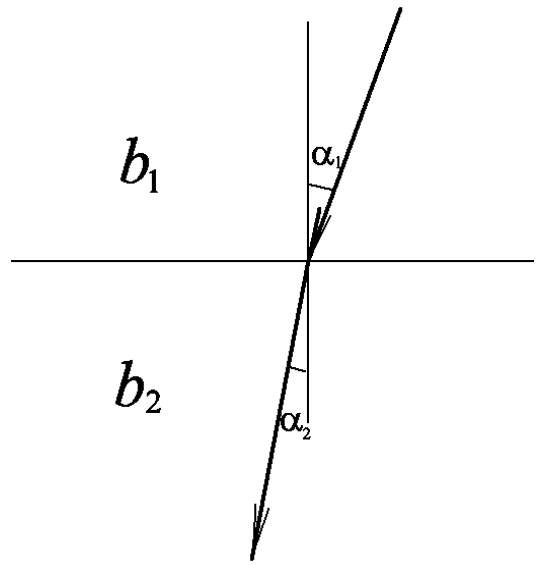

Figure 2.1. Wave ray refraction at the interface between two media

Thus, in a medium where the conductivity (wave propagation velocity) b varies only along one spatial variable (for instance, $b(y)$), the wave ray inclination from the direction of a change in the conductivity α changes according to the formula

$$\frac{\sin(\alpha(y))}{b(y)} = const1 = \frac{\sin(\alpha_0)}{b(y_0)} \quad . \tag{2.2}$$

Here α_0 is the initial incidence angle of the wave ray with respect to the vertical at the point $y=y_0$. In the case of a sloping bottom the medium conductivity (tsunami wave propagation velocity) is determined by Lagrange's formula (1.3), which for a sloping bottom has the following form:

$$b(y) = \sqrt{g \times y \times tg(\beta)} \quad , \tag{2.3}$$

where g is the acceleration of gravity, β is the bottom slope angle, and y is the distance to the coastline, where $y = 0$. Hence, the relation between the inclination angle (optimal trajectory) and the distance to the coast has the form

$$\sin^2(\alpha) = y \times \left(\frac{g \times tg(\beta) \times \sin^2(\alpha_0)}{b(y_0)^2} \right) = y \times \left(\frac{\sin^2(\alpha_0)}{y_0} \right) = const2 \times y \quad , \tag{2.4}$$

where the value of $const2$ is determined from the ray inclination at the distance y_0 to the coast ($0x$–axis). If α is assumed to be the parameter on which y depends, then from (2.4) we have

$$dy = \frac{2\sin(\alpha)\cos(\alpha)}{const2} \times d\alpha \quad . \tag{2.5}$$

Since ($\pi/2 - \alpha$) is the wave ray inclination angle (the graphic of the function $y(x)$) with respect to the horizontal direction, then according to the definition of the derivative of a function of one variable the following equality is valid:

$$\frac{dy}{dx} = \frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos\left(\frac{\pi}{2} - \alpha\right)} = \frac{\cos(\alpha)}{\sin(\alpha)} \quad ,$$

or

$$dx = dy\frac{\sin(\alpha)}{\cos(\alpha)} \quad . \tag{2.6}$$

From (2.5) and (2.6) we have

$$dx = \frac{2\sin(\alpha)\times\cos(\alpha)}{const2} \times \frac{\sin(\alpha)}{\cos(\alpha)} \times d\alpha = \frac{2\sin^2(\alpha)}{const2} d\alpha \quad . \tag{2.7}$$

Assuming that x and y depend on the parameter $u = 2\alpha$ and using trigonometric formulas for the sine and cosine of a double angle, we obtain the following formulas from (2.5) and (2.7):

$$dy = \frac{\sin(u)}{2\times const2} \times du \quad , \quad dx = \frac{1-\cos(u)}{2\times const2} \times du \tag{2.8}$$

Integrating equalities (2.8), we obtain the following equations for the wave ray trajectory in the parametric form:

$$x(u) = r(u - \sin(u)) + C_2$$

$$y(u) = r(C_3 - \cos(u)) \quad u \in [0, 2\pi], \tag{2.9}$$

$$r = 1/2 \times const2 = y_0 / 2\sin^2(\alpha_0) \quad .$$

This is a parametric form of the cycloid equation. Here the constants C_2 and C_3 are determined when the cycloid passes through the origin of coordinates. At the point (0,0) the parameter u is zero. This follows from Snell's law for the bottom in question (2.4). At $y = 0$, the angle α and the parameter $u = 2\alpha$ are zero. Hence, $C_2 = 0$, $C_3 = 1$. Finally, the wave ray equations in the parametric form are presented as

$$x(u) = r(u - \sin(u))$$
$$y(u) = r(1 - \cos(u)) \qquad u \in [0, 2\pi] \tag{2.10}$$

When the equations are written in this form, the parameter u is the doubled ray inclination angle with respect to the vertical direction and the coefficient r is determined in each specific case. In this boundary value problem for a wave ray, the parameter r is determined from the condition of ray passing through the point (x_1, y_1); here the second point is the origin of coordinates. For a wave ray which at offshore distance y_1 is directed at an angle α_1 with respect to the normal to the coastline (0x–axis), from (2.4) and (2.9) it follows that

$$r = \frac{y_1}{2 \times \sin(\alpha_1)} \tag{2.11}$$

Thus, we have obtained the equations describing the wave ray propagation above a sloping bottom using the laws of ray motion in a varying conductivity medium.

3. DETERMINATION OF A WAVE-FRONT LINE ABOVE THE BOTTOM SLOPE

For some model shapes of bottom, distributions of wave amplitudes (heights) can be found analytically. Consider, for example, a coastal area where the depth linearly increases with distance to the coast with a model source of tsunami in the form of a circle of radius R_0 with the center at the distance y_{00} from a straight coastline. In Fig. 3.1, this line coincides with the axis Ox ($y = 0$). In Section 2, the wave ray trajectory over a sloping bottom (as in the case in question) has already been found.

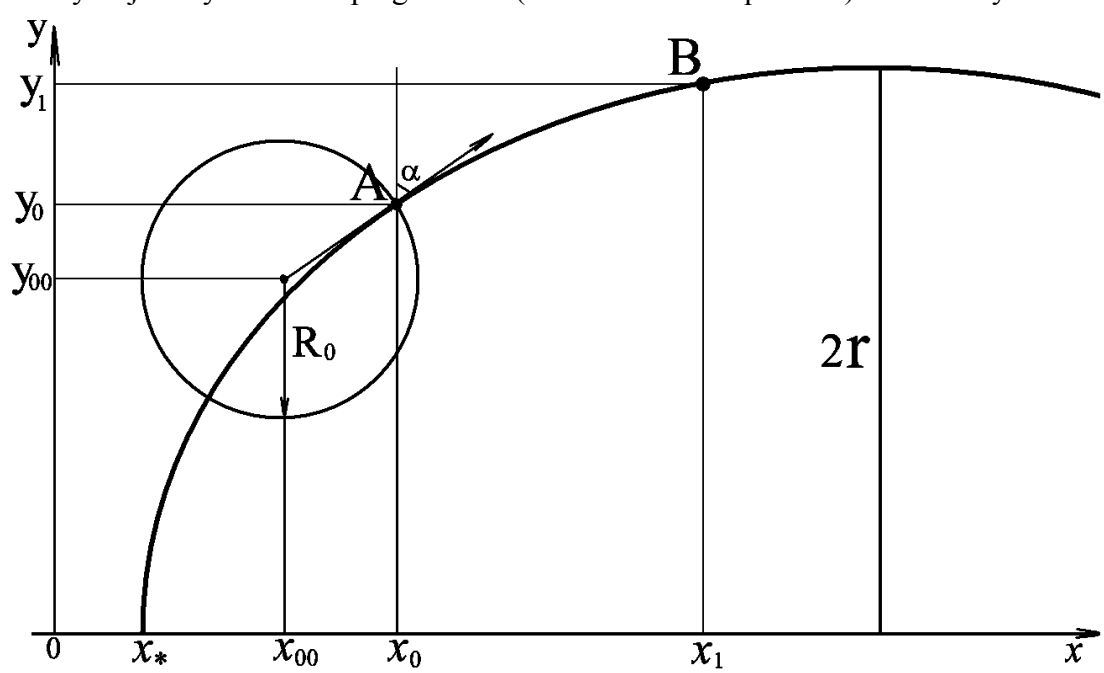

*Figure 3.1. A wave ray trajectory over a sloping bottom with inclination α with respect to the ordinate axis at the point **A***

If the depth is given by the formula

$$D(x, y) = a \times y \qquad (a > 0) \quad , \tag{3.1}$$

then the wave ray trajectory (as has been found) has the form of a cycloid whose equations in the parametric form have the following form:

$$\begin{aligned} x &= r(u - \sin(u)) + x_* \\ y &= r(1 - \cos(u)) \end{aligned} \quad , \tag{3.2}$$

Here x_* is the abscissa of reaching the coast by the cycloid, and r is its radius determined from its passage through given points or through its inclination angle at a certain distance from the coast. The parameter u is equal to the doubled inclination angle of a ray with the vertical. All parameters of the cycloid can be easily determined if the angle between the vertical and the tangent to the cycloid is known at some cycloid point. Thus, we construct a wave ray moving, at the angle α, from a point (x_0, y_0) located at the boundary of a circular tsunami source with radius R_0. Coordinates of the tsunami source center are (x_{00}, y_{00}) (see Fig. 3.1). The exit point coordinates are

$$\begin{aligned} x_0 &= x_{00} + R_0 \times \sin(\alpha) \\ y_0 &= y_{00} + R_0 \times \cos(\alpha) \end{aligned} \quad , \tag{3.3}$$

The situation when the cycloid from the point **A**(x_0, y_0) goes up is somewhat different from the situation when it goes down. First consider the case when the angle α is in the interval (0, π/2) (see Fig. 3.1.) The radius of the cycloid can be easily calculated from formulas (3.2):

$$r = \frac{y_0}{1 - \cos(2\alpha)} \tag{3.4}$$

Let us obtain a formula for the time duration of the wave motion along this cycloid from the point (x_0, y_0) to the point (x_1, y_1). At the exit point **A**(x_0, y_0), the parameter u is

$$u_0 = 2\alpha \quad . \tag{3.5}$$

This cycloid starts at the coastline point (x_*,0) (Fig. 3.1):

$$x_* = x_{00} + R_0 \times \sin(\alpha) - \frac{y_0}{1 - \cos(2\alpha)}\left(2\alpha - \sin(2\alpha)\right) \quad . \tag{3.6}$$

Obtaining the radius from formula (3.4) and expressing the wave velocity in terms of the bottom topography (3.1) and Lagrange's formula (1.3), we can determine the time duration of wave motion along this cycloid from the point **A**(x_0, y_0) to the point **B**(x_1, y_1) in the form of the Fermat integral:

$$T = \int_A^B \frac{ds}{\sqrt{g \times a \times y}} = \int_A^B \frac{\sqrt{(r - r \times \cos(u))^2 + (-r \times \sin(u))^2}\, du}{\sqrt{g \times a \times r(1 - \cos(u))}} =$$

$$= \int_A^B \frac{r\sqrt{2(1 - \cos(u))}}{\sqrt{g \times a \times r(1 - \cos(u))}} du = (u_B - u_A)\sqrt{\frac{2r}{g \times a}} \quad , \qquad (3.7)$$

where u_B and u_A are values of the cycloid parameter at the points **B** and **A**, respectively. The value at the point **B** can be easily found from (3.2):

$$u_B = \arccos\left(1 - \frac{y_1}{r}\right) \quad . \qquad (3.8)$$

As was mentioned above, the parameter u_A is equal to 2α. Finally, we obtain an expression for the time duration of wave motion from the point **A**(x_0, y_0) to the point **B**(x_1, y_1) as follows:

$$T = \sqrt{\frac{2r}{g \times a}} \times \left(\arccos\left(1 - \frac{y_1}{r}\right) - 2\alpha\right) \quad , 0 < \alpha < \pi/2 \ . \qquad (3.9)$$

Here the radius r is determined from (3.4). Now let us formulate the problem in a different way. Let the wave propagation time from the points of an initial circular front be known. We find the coordinates of the points along the corresponding wave rays, where the wave will arrive at the time T. For this, we first express u_B from equation (3.7) in terms of the parameters of the cycloid and the time T. As a result, we have

$$u_B = u_A + T \times \sqrt{\frac{g \times a}{2r}} = 2\alpha + T \times \sqrt{\frac{g \times a}{2r}} = 2\alpha + \sqrt{\frac{g \times a \times (1 - \cos(\alpha))}{2y_0}} \quad , \qquad (3.10)$$

Now, from formulas (3.2) it is easy to find the coordinates of the wave front point in the wave ray in question at the time T:

$$x_1 = \frac{y_0}{1 - \cos(2\alpha)}\left(2\alpha + \sqrt{\frac{g \times a \times (1 - \cos(\alpha))}{2y_0}} - \sin\left(2\alpha + \sqrt{\frac{g \times a \times (1 - \cos(\alpha))}{2y_0}}\right)\right) + x_* \quad , \qquad (3.11)$$

$$y_1 = \frac{y_0}{1-\cos(2\alpha)}\left(1-\cos\left(2\alpha + \sqrt{\frac{g \times a \times (1-\cos(\alpha))}{2y_0}}\right)\right), \quad 0 < \alpha < \pi/2. \tag{3.12}$$

If the exit angle α of the wave ray from the point (x_0, y_0) is in the interval $\pi/2 > \alpha > \pi$, formulas (3.11) and (3.12) can be slightly changed and written down as

$$x_1 = \frac{y_0}{1-\cos(2\alpha)}\left(2\alpha - \sqrt{\frac{g \times a \times (1-\cos(\alpha))}{2y_0}} - \sin\left(2\alpha - \sqrt{\frac{g \times a \times (1-\cos(\alpha))}{2y_0}}\right)\right) + x_*, \tag{3.13}$$

$$y_1 = \frac{y_0}{1-\cos(2\alpha)}\left(1-\cos\left(2\alpha - \sqrt{\frac{g \times a \times (1-\cos(\alpha))}{2y_0}}\right)\right). \tag{3.14}$$

When the exit angle α is equal to 0 or π the wave ray comes straight off or toward the coastline. Along these straight lines the wave travel time can be expressed as the route length divided by the arithmetic mean of wave velocities c_1 and c_2 at the start and destination points

$$T = \frac{y_1 - y_0}{(c_1 + c_0)/2} = \frac{2(y_1 - y_0)}{\sqrt{g \times a \times y_1} + \sqrt{g \times a \times y_0}}, \quad y_1 > y_0. \tag{3.15}$$

In order to obtain the ordinate y_1 of the wave isochrone along these two rays it is necessary to solve the following equation

$$f \times \left(\sqrt{y_1} + \sqrt{y_0}\right) = y_1 - y_0 = \left(\sqrt{y_1} + \sqrt{y_0}\right)\left(\sqrt{y_1} - \sqrt{y_0}\right), \quad f = \frac{T\sqrt{ga}}{2} \tag{3.16}$$

Finally, the formulas for the solutions y_1 are as follows:

$$y_1 = \left(\sqrt{y_0} + f\right), \quad y_1 > y_0 \text{ and } y_1 = \left(\sqrt{y_0} - f\right), \quad y_1 < y_0. \tag{3.17}$$

Thus, we have obtained the coordinates of the destination point versus time T and the angle α. Now, with formulas (3.9) and (3.10) we can find the wave front location by fixing the time T and taking the values α in the entire interval ($-\pi$, $+\pi$) with a sufficiently small step $\Delta\alpha$. It should be noted that in the case of a circular initial tsunami front the value of x_* and the coordinates of the wave ray exit points (x_0, y_0) vary according to formulas (3.6) and (3.3). Figure 3.2 presents the wave rays from the circled source boundary using formulas (3.11)-(3.14) and (3.17).

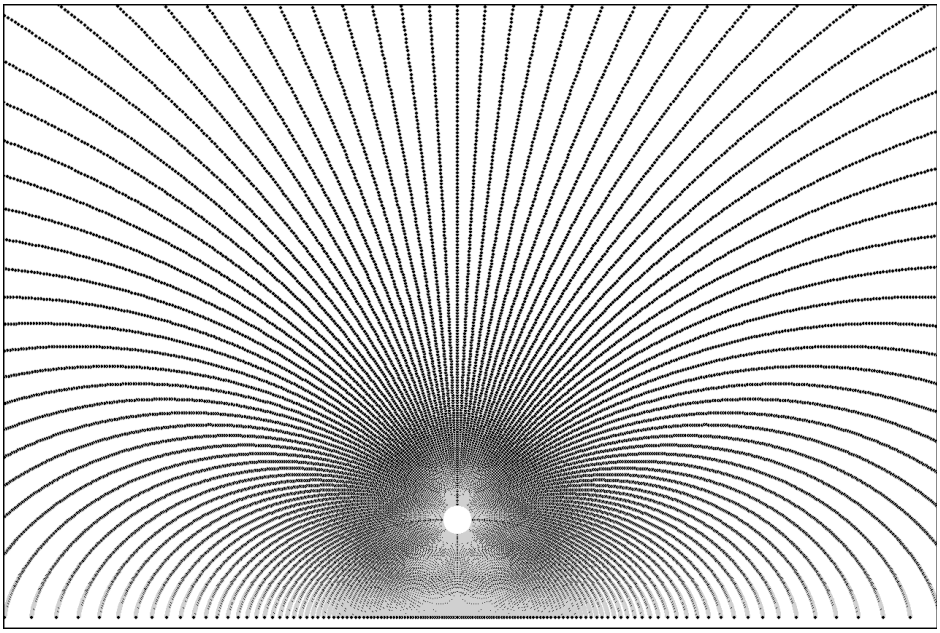

Figure 3.2. Traces of wave rays from the round source boundary above the bottom slope

If we draw the lines connecting points along wave the rays corresponding to the same time instance, then we will obtain tsunami isochrones. For example, Figure 3.3 shows locations of the wave front within 5 minutes interval.

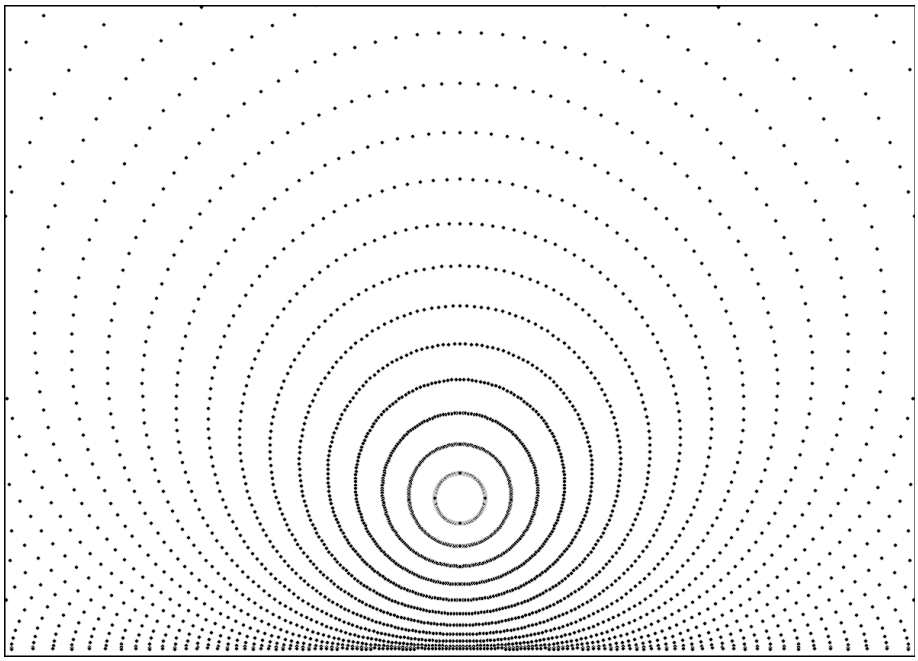

Figure 3.3. The shapes of tsunami isochrones from the round-shaped source within 5 minutes interval

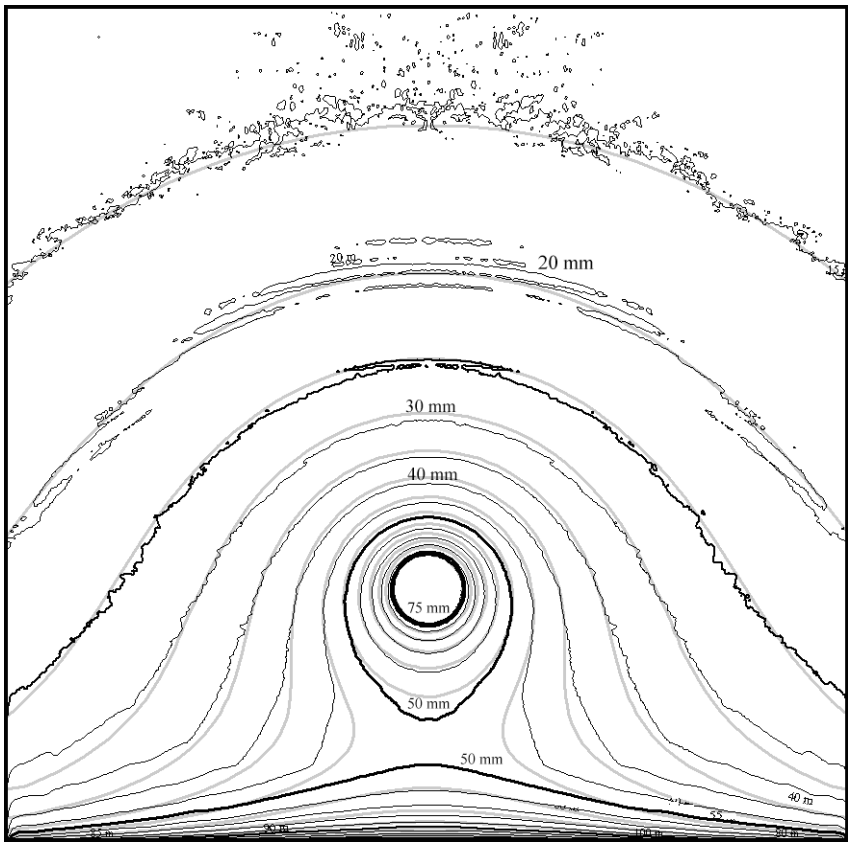

Figure 4.1. Comparative location of isolines of tsunami height maxima calculated by the numerical shallow water model (Titov, Gonzalez, 1997) (black color) and by the ray approximation (grey color)

4. ESTIMATION OF THE TSUNAMI WAVE HEIGHT WITHIN THE WAVE-RAY APPROXIMATION

If we want to estimate the wave height at the point (x_1, y_1) it is necessary to determine the distance between two points. The first one is the point (x_1, y_1), where the wave going along the ray exiting the point (x_{00}+R0 sin α, y_{00}+R_0 cos α) at the angle α (fig. 3.1) arrives at the time T. The second one is the point (x_2, y_2), where a tsunami wave arrives at the same time moment going along the wave ray exiting the point (x_{00}+R_0 sin(α+$\Delta\alpha$), y_{00}+R_0 cos(α+$\Delta\alpha$)) at angle $\alpha + \Delta\alpha$. With formulas (1.19) and (1.14), the coefficient of wave attenuation due to changing the ray tube width and depth is calculated. Doing this for various values of the time T and the directions of wave rays, we obtain the wave attenuation distribution over the entire area of points which can be reached by the wave rays from the initial wave front points. To verify the solution obtained, the numerical simulation of tsunami wave propagation was carried out using the differential shallow-water model with a package called MOST (Titov, Gonzalez, 1997). In this numerical experiment, the center of a circular 2 m height source, 40 km in radius, was located at a distance of 300 km from the coast. This source formed a 75-cm high circular wave at a distance of 43.6 km from the center. This initial front was taken as initial conditions to

calculate the amplitudes with the ray model. In Fig. 3.4, isolines of the tsunami wave height maxima in a 1000 × 1000 km coastal area with a sloping bottom obtained from formulas (3.11)–(3.14) and (3.2) are shown by grey color. For comparison, isolines of the wave height maxima obtained by numerical solution of the same problem with the shallow water model having the same initial values are shown by black color. In both cases, the levels of isolines (whose height is shown in cm), were taken with a spacing of 5 cm. The figure 3.4 shows that the distributions of amplitudes obtained by the two different methods mostly coincide. An exception is a coastal band where, in contrast to the ray approximation, the numerical implementation of the differential shallow water model uses the boundary conditions of the full reflection at the coastline. This approximately doubles the height of the wave that arrives there.

CONCLUSIONS

The height of a propagating tsunami wave versus depth and refraction above an uneven bottom has been estimated from the differential shallow-water equations. The exact wave ray trajectory and tsunami isochrones above a sloping bottom has been found. A comparison of the results obtained by the ray method and with the shallow water model has been made. It shows that with a numerical method based on the ray approximation not only the arrival times of tsunami waves at different points, but also the wave heights can be estimated. These benchmark solutions can be used for testing the numerical methods in the tsunami modeling.

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